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Random packing by matroid bases and triangles

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Abstract

Let M be a matroid on a finite set $E(M)$. Then M is packable by bases if $E(M)$ is the disjoint union of bases. A partial packing of M is a collection of disjoint bases whose union is a proper subset of $E(M)$. M is a randomly packable by bases if every partial packing can be extended to a packing of M. This paper determines the structure of the matroids that are randomly packable by bases. It also gives a characterization, in terms of forbidden restrictions, of the simple matroids that are randomly packable by 3-circuits.

1. Introduction

Let M be a matroid on a finite set $E(M)$. For a given matroid N, we say M is *N-packable* if $E(M)$ has a partition $\{N_1, N_2, \ldots, N_k\}$ such that the restriction $M|N_i$ is isomorphic to N for all i in $\{1, 2, \ldots, k\}$. Such a partition will be denoted by $\mathscr{P}$, and we say that $\mathscr{P}$ is an *N-packing* of M. A *partial N-packing* of M is a collection of disjoint sets $\{N_1, N_2, \ldots, N_l\}$ such that $\bigcup_i N_i$ is a proper subset of $E(M)$ and $M|N_i$ is isomorphic to N for all i in $\{1, 2, \ldots, l\}$. The matroid M is *randomly N-packable* if every partial N-packing is contained in an N-packing of M.

A graph G is *packable* by a graph H if its edges can be partitioned into copies of H. The graph G is *randomly H-packable* if deleting the edges of any H-packable subgraph from G leaves an H-packable graph. The notion of random packing for graphs was introduced by Ruiz [9]. In that paper, he gave a list of all randomly H-packable graphs in the two cases where H consists of two independent edges and where H consists of a path of length two. Various generalizations and results were given by Barrientos et al. [1] and by Beineke et al [2]. In particular Beineke et al. [3] showed that a simple graph is randomly packable by triangles if and only if it does not contain the graph shown in Fig. 1 as a subgraph.

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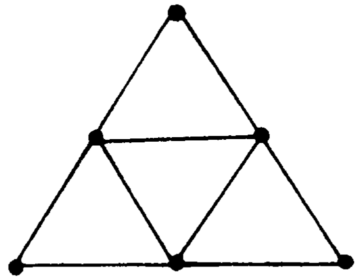


Fig. 1.

In this paper, we give a structural characterization of the matroids that are randomly packable by bases. We also characterize, in terms of forbidden restrictions, the simple matroids that are randomly packable by 3-circuits.

The matroid terminology used here will follow Oxley [8]. If M is a matroid, then its ground set and rank will be denoted by $E(M)$ and $r(M)$, respectively. If $X \subseteq E(M)$, then the restriction of M to X will be denoted by $M|X$. The deletion of X from M will be denoted by $M\backslash X$. A 3-circuit will be called a *triangle*. The dual of a matroid M will be denoted by M^*. The property that a circuit and a cocircuit cannot have exactly one element in common will be called *orthogonality*. The uniform matroid of rank r on an n-element set will be denoted by $U_{r,n}$.

Let B be a base of M and e be an element of $E(M) - B$. Then $C(e, B)$ denotes the fundamental circuit of e with respect to B. If X is a circuit-hyperplane of M, then we can obtain a new matroid M' by *relaxing* X; the bases of M' are the bases of M together with X.

We close this section with a result which shows the hereditary nature of random packing. The proof is omitted since it is identical to that given in [2] of the corresponding result for graphs.

Lemma 1.1. *Let M be a randomly N-packable matroid and X be a subset of $E(M)$. If $M|X$ is N-packable, then it is randomly so.*

2. Random packing by bases

In this section we give a characterization of the matroids that are randomly packable by bases. First recall that a matroid is *identically self-dual* (*ISD*) if its sets of bases and cobases coincide. These matroids were investigated by Bondy and Welsh [4]. They showed how to construct a large number of connected, transversal ISD matroids. Moreover, Graver [7] showed that there are connected, binary ISD matroid of rank r for every positive integer r not equal to 2, 3 or 5. It is clear that every

ISD matroid is randomly base-packable. The next theorem shows that, with one exception, the converse is also true for connected matroids.

Theorem 2.1. *Let M be a connected matroid having rank r. If M is randomly base-packable, then M is ISD or $M \cong U_{r,kr}$ for some positive integer k.*

We remark here that if a randomly base-packable matroid is not connected, then all of its components have the same number of bases. Thus, by Theorem 2.1, such a matroid M is a direct sum of ISD matroids and uniform matroids of the form $U_{r,kr}$ where all the components of M have the same number of bases.

The following two lemmas are needed in the proof of Theorem 2.1. The first of these appears in [6].

Lemma 2.2. *Let B be a base of a matroid M and suppose that M is not isomorphic to $U_{1,1}$. Then M is connected if and only if $B \subseteq \bigcup\{C(e,B)\colon e \in E(M) - B\}$ and, for any partition $\{X, Y\}$ of $E(M) - B$, there is an element x in X and an element y in Y such that $C(x,B) \cap C(y,B) \neq \emptyset$.*

Lemma 2.3. *Let M be a randomly base-packable matroid. Then every circuit of M is contained in a cocircuit.*

Proof. Let C be a circuit of M and e be an element of C. Then $C - e$ is contained in a base B_1 of M. Moreover, it is clear that if x is any element of $C - e$, then $(B_1 - x) \cup e$ is a base of M. As M is randomly packable, e is in a base B_2 of M such that $B_1 \cap B_2 = \emptyset$. Now the partial packing $\{B_1, B_2\}$ is contained in a packing $\{B_1, B_2, \ldots, B_m\}$ of M. The $\bigcup_{i \neq 2} B_i$ is a cobase of M. Therefore e is in a cocircuit C^* of M such that $C^* \cap B_2 = \{e\}$. Now suppose that x is an element of $C - e$ such that x is not in C^*. Then, since $\{(B_1 - x) \cup e, B_3, \ldots, B_m\}$ is a partial packing of M, we get that $(B_2 - e) \cup x$ is a base of M. This is a contradiction since $E(M) - ((B_2 - e) \cup x)$ would be a cobase of M containing C^*. Therefore every element of C is contained in C^*. $\square$

The proof of Theorem 2.1. If M is packable by one base, then, since it is connected, M is isomorphic to $U_{1,1}$. If M is packable by two bases, then it is ISD. So assume that M is packable by more than two bases. We shall show that, for some integer k exceeding 2, M is isomorphic to $U_{r,kr}$ by showing that every circuit has $r + 1$ elements. Assume, to the contrary, that M has a circuit C with fewer than $r + 1$ elements. Let e be an element of C. Then $C - e$ is contained in a base B_1, and e is in a base B_2 of M such that $B_1 \cap B_2 = \emptyset$. Thus, by Lemma 2.3 and its proof, there is a cocircuit C^* of M such that $C \subseteq C^*$ and $C^* \cap B_2 = \{e\}$. Note that, since $|C| < r + 1$, $B_1 - C \neq \emptyset$. Let $\{B_1, B_2, \ldots, B_m\}$ be a packing $\mathscr{P}$ of M. $\square$

Lemma 2.4. *Suppose $i \neq 2$. Let f be an element of $B_i \cap C^*$. Then $(B_2 - e) \cup f$ is a base of M.*

Proof. As f is in C^* and $C^* \cap (B_2 - e) = \emptyset$, orthogonality implies that there is no circuit that both contains f and is contained in $(B_2 - e) \cup f$. $\square$

Lemma 2.5. *Suppose $i \geqslant 3$. Let g be an element of $B_1 - C$. Then $C(g, B_i) \cap (C^* \cap B_i) = \emptyset$.*

Proof. Suppose $f \in C(g, B_i) \cap (C^* \cap B_i)$. Then $(B_i - f) \cup g$ is a base of M. Also, by Lemma 2.4, $(B_2 - e) \cup f$ is a base of M. Then since $\{(B_2 - e) \cup f, (B_i - f) \cup g\} \cup (\mathscr{P} - (B_1, B_2, B_i\})$ is partial packing of M, we get that $(B_1 - g) \cup e$ is a base of M. This is a contradiction since C is contained in $(B_1 - g) \cup e$. $\square$

Lemma 2.6. $B_1 - C^* \neq \emptyset$.

Proof. This follows from orthogonality and Lemma 2.5. $\square$

Lemma 2.7. *Suppose $i \geqslant 3$. Let g be an element of $B_1 - C^*$. Then there is an element f in $B_i - C^*$ such that $(B_1 - g) \cup f$ and $(B_i - f) \cup g$ are bases of M.*

Proof. Using symmetric base exchange, there is an element f in B_i such that $(B_1 - g) \cup f$ and $(B_i - f) \cup g$ are bases of M. Clearly, $f \in C(g, B_i)$. Therefore, by Lemma 2.5, f is in $B_i - C^*$. $\square$

Lemma 2.8. *Suppose $i \geqslant 3$. If f is an element of $B_i \cap C^*$, then there is an element g of $B_1 \cap C^*$ such that $(B_1 - g) \cup f$ and $(B_i - f) \cup g$ are bases of M.*

Proof. By symmetric base exchange, there is an element g of B_1 such that $(B_1 - g) \cup f$ and $(B_i - f) \cup g$ are bases of M. It remains to show that $g \in B_1 \cap C^*$. If g were in $B_1 - C^*$, then, since $B_1 - C^* \subseteq B_1 - C$, Lemma 2.5 would imply that $C(g, B_i)$ is contained in $(B_i - f) \cup g$, a contradiction. Therefore g is in $B_1 \cap C^*$. $\square$

Lemma 2.9. *Suppose $i \geqslant 3$. Then $C(e, B_i) \subseteq C^*$.*

Proof. The proof is similar to that of Lemma 2.3 and is omitted. $\square$

Now consider the base B_2. Then $\{C^* - e, E(M) - (C^* \cup B_2)\}$ is a partition of $E(M) - B_2$. Therefore, by Lemma 2.2, there is an element x in $C^* - e$ and an element y in $E(M) - (C^* \cup B_2)$ such that $C(x, B_2) \cap C(y, B_2) \neq \emptyset$. By Lemmas 2.7 and 2.8, we may assume that $x \in B_1$ and $y \in B_i$ for some $i \geqslant 3$. Let z be an element of $C(x, B_2) \cap C(y, B_2)$. Then, by orthogonality of $C(y, B_2)$ with C^*, the elements z and e are distinct. Now $(B_2 - z) \cup x$ is a base of M. Then, by using the partial packing $\{(B_2 - z) \cup x\} \cup (\mathscr{P} - \{B_1, B_2\})$, we get that $(B_1 - x) \cup z$ is a base of M.

Now let $B = (B_2 - z) \cup x$. Then $x \in C(y, B)$, otherwise y would be in two fundamental circuits with respect to B_2. Thus, by orthogonality of $C(y, B)$ with $C^*, e \in C(y, B)$. Therefore $(B - e) \cup y$ is a base of M. Now, by using the partial packing $\{(B_1 - x) \cup z), (B - e) \cup y\} \cup \{\mathscr{P} - \{B_1, B_2, B_i\}\}$, we get that $(B_i - y) \cup e$ is a base of M. This is a contradiction since, by Lemma 2.9, $C(e, B_i)$ is contained in $B_i \cap C^*$. Thus $|C| = r + 1$. Hence M is isomorphic to $U_{r,kr}$, where k is an integer exceeding two. $\square$

3. Random packing by triangles

In this section we give a characterization, in terms of forbidden restrictions, of the simple matroids that are randomly triangle-packable. A matroid F is a *forbidden restriction* of random packability by a given matroid N if

(1) F is N-packable but is not randomly so; and

(2) (minimality) every N-packable proper restriction of F is randomly N-packable.

We remark here that, by Lemma 1.1, an N-packable matroid is randomly N-packable if and only if it does not contain a forbidden restriction.

The cycle matroid F_1 of the graph shown in Fig. 1 is a forbidden restriction of random packability by triangles. So are the four matroids shown in Table 1.

Other forbidden restrictions can be obtained from the ternary affine plane $AG(2,3)$, shown in Fig. 2, by relaxing some of its circuit-hyperplanes.

The twelve triangles of $AG(2,3)$ can be uniquely partitioned into four packings. Let $\mathscr{R}$ be the set of nonisomorphic matroids that are obtained from $AG(2,3)$ by relaxing exactly one or exactly two triangles of one packing and relaxing no triangles in another packing, while each triangle of the remaining two packings may or may not be relaxed. Then it is easy to see that each member of $\mathscr{R}$ is a forbidden restriction. In fact, we shall show that, for simple matroids, $\{F_1, F_2, F_3, F_4, F_5\} \cup \mathscr{R}$ is a complete set of forbidden restrictions. To prove this we need the following two results which appear in [5]. The proofs are omitted since they are identical to those given in [5] of the corresponding results for graphs.

Theorem 3.1. *Let F be a forbidden restriction for random packability by some matroid N. Let $\{N_1, N_2, \ldots, N_k\}$ be a packing of F. Then there is a subset X of $E(F)$ such that $F|X$ is isomorphic to N and both* (i) *and* (ii) *hold.*

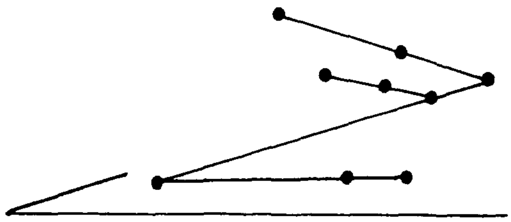
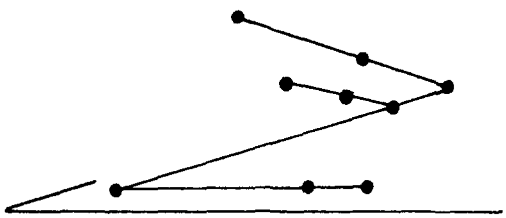
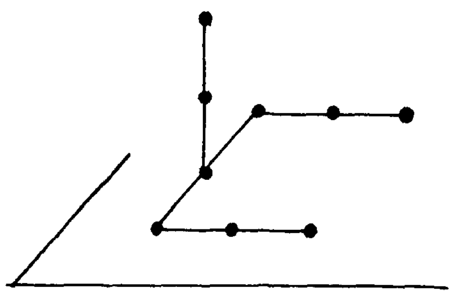
(i) *For all i in $\{1, 2, \ldots, k\}$, $X \cap E(N_i) \neq \emptyset$.*

(ii) *$F \backslash X$ is not N-packable.*

Corollary 3.2. *F has at most $|E(N)|^2$ elements.*

Theorem 3.3. *For simple matroids, $\{F_1, F_2, F_3, F_4, F_5\} \cup \mathscr{R}$ is a complete set of forbidden restrictions for random packability by triangles.*

Table 1

Matroid	Geometric representation	Rank	Remarks
F_2		3	
F_3		4	No three lines in a plane
F_4		4	No three lines in a plane
F_5		4	Three lines in a plane.

Proof. Let F be a forbidden restriction. Then, by Corollary 3.2, F has six or nine elements. First suppose $|E(F)| = 6$. Let $\{T_1, T_2\}$ be a packing of F. Then, by Theorem 3.1, there is a triangle T of F such that T intersects T_1 and T_2, and $F \setminus T$ is not triangle-packable. Now it is easy to check that F is isomorphic to F_2 shown in Table 1.

Now suppose $|E(F)| = 9$. Let $\{T_1, T_2, T_3\}$ be a packing of F and T be a triangle as in Theorem 3.1. As $r(F) \leqslant 6$ and T intersects T_1, T_2 and T_3, $r(F) \leqslant 5$. Suppose $r(F) = 5$.

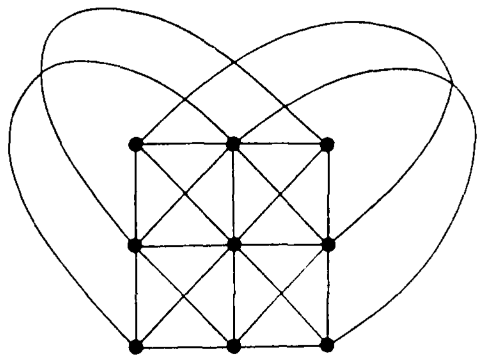


Fig. 2.

Then, since $F \setminus T_1$ is the disjoint union of two triangles and since $T_1 \cap T \neq \emptyset$, $r(F \setminus T_1) = 4$. Now it is easy to see that F is isomorphic to F_1.

Now to deal with the cases where $r(F) \leqslant 4$, we need the following lemma.

Lemma 3.4. *Suppose $r(F) \leqslant 4$. If S is a triangle of F such that S is not in $\{T_1, T_2, T_3, T\}$, then S intersects each of T_1, T_2 and T_3. Moreover, $|S \cap T| \leqslant 1$.*

Proof. If, for some i in $\{1,2,3\}$, $|S \cap T_i| = 2$, then $F|(S \cup T_i \cup T)$ is isomorphic to F_2. This contradicts the minimality of F. The proof of the last conclusion is similar. □

Now suppose $r(F) = 4$. We consider the following two cases.

Case 1. Suppose, for all i in $\{1,2,3\}$, $r(F \setminus T_i) = 4$. If T is the only triangle of F distinct from T_1, T_2 and T_3, then, since T intersects each of T_1, T_2, and T_3, F is isomorphic to F_3. Now suppose F has a triangle S such that S is not in $\{T_1, T_2, T_3, T\}$. Then, by Lemma 3.4, $|S \cap T| \leqslant 1$. Next we show that $S \cap T = \emptyset$. Assume, to the contrary, that $|S \cap T| = 1$. Then, without loss of generality, we may assume $S \cap T \cap T_1 \neq \emptyset$. Note that, for i in $\{2,3\}$, $S \cap T \cap T_i = \emptyset$. Then, since, for i in $\{2,3\}$, $S \cap T_i$ and $T \cap T_i$ are nonempty and since $S \cap T_i \neq T \cap T_i$, we must have $r(S \cup T \cup T_i) = 3$. Thus $r(T_2 \cup T_3) = 3$. This contradicts the assumption that $r(F \setminus T_1) = 4$. Therefore $S \cap T = \emptyset$.

Now if S is the only triangle of F such that S is not in $\{T_1, T_2, T_3, T\}$, then F is isomorphic of F_4. Moreover, if S' is a triangle of F distinct from S, then F is isomorphic to $M^*(K_{3,3})$, which is shown in Fig. 3. This is not possible since $M^*(K_{3,3})$ is randomly packable by triangles.

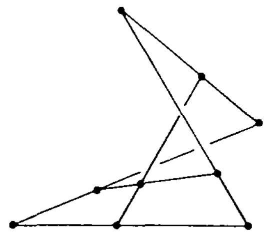


Fig. 3.

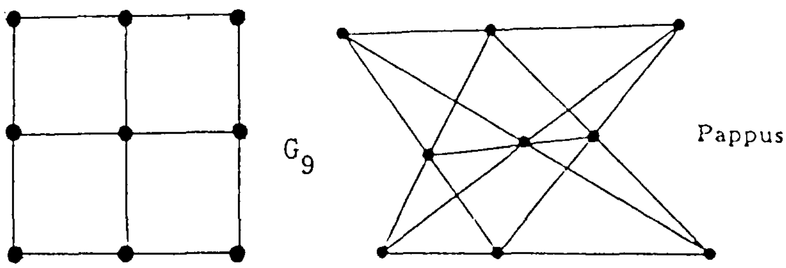


Fig. 4.

Case 2. Suppose that, for some i in $\{1,2,3\}$, $r(F \backslash T_i) = 3$. Then, by Lemma 3.4 and the fact that $r(F) = 4$, T is the only triangle of F distinct from T_1, T_2 and T_3. Thus F is isomorphic to F_5.

Now suppose $r(F) = 3$. Then, using Lemma 3.4, it is not difficult to see that F is a member of $\mathscr{R}$. $\square$

The last result of this section describes the local structure of simple, randomly triangle-packable matroids around two intersecting triangles. The matroid G_9 is obtained by relaxing all the triangles of two packings of AG(2, 3). The Pappus matroid is obtained by relaxing all the triangles of a single packing of AG(2, 3). Geometric representations of these two matroids are shown in Fig. 4.

Theorem 3.5. *Let M be a simple randomly triangle-packable matroid. If M has two intersecting triangles, then it has a restriction containing these two triangles that is isomorphic to $U_{2,6}, M^*(K_{3,3}), G_9$, the Pappus matroid, or* AG(2, 3).

Proof. We omit the straightforward proof which uses Lemma 1.1 and Theorem 3.3. $\square$

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